Simple Minimal Informationally Complete Measurements for Qudits

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(Dated: August 2005)

Simple minimal but informationally complete positive operator-valued measures are constructed out of the expectation-value representation for qudits. Upon suitable modification, the procedure transforms any set of d^2 linearly independent hermitean operators into such an observable. Minor changes in the construction lead to closed-form expressions for informationally complete positive measures in the spaces $\mathbb{C}^d$.

I. INTRODUCTION

The very idea to implement information on quantum systems and to subsequently process it [1] requires to initially *prepare* a particular quantum state $\hat{\rho}_{in}$, to *verify* the preparation procedure and to *identify* the final state $\hat{\rho}_{out}$ produced by the quantum dynamics. Since an unknown quantum state cannot be determined unambiguously by a single measurement and no copies of the state can be made, one needs to resort to repeated measurements of identically prepared states. It is thus desirable to design quantum mechanical algorithms in such a way that only a small number of *known* final states can arise: it is easier then to extract the desired information with high probability from a few runs as in Shor's algorithm [2], for example. Both the verification of a state and its identification are instances of *state reconstruction* or *estimation* [3].

A useful measure for the reliability of a measurement procedure to determine an unknown quantum state is given by its *fidelity F*, the mean overlap of the reconstructed state with the exact state [4]. Perfect fidelity usually requires an infinite supply of the unknown state. This might sound unrealistic from an experimental point of view. The theoretical possibility to achieve $F = 1$ for an arbitrary input state with a given set of measurements says that the measurements are *complete*. In the following, the focus will be on sets of hermitean operators which allow one, in principle, to perfectly reconstruct an unknown state described by a density operator $\hat{\rho}$ in a d-dimensional Hilbert space $\mathbb{C}^d$. Many such bases for observables are known [5, 6], and in some cases they have been combined into what is called *minimal informationally complete positive operator-valued measures* (cf. below) which is important from a conceptual point of view.

This contribution will strengthen the links between state reconstruction, minimal complete sets of hermitean operators, and positive operator-valued measures. Firstly, it argues that one can extract minimal informationally complete measures from the *expectation-value representation* of quantum states in finite-dimensional Hilbert spaces [7]. Secondly, this approach will be adapted to construct such measures out of any set of d^2 linearly independent hermitean operators on $\mathbb{C}^d$.

II. POSITIVE OPERATOR-VALUED MEASURES

A. Properties and Examples

Positive operator-valued measures [8] correspond to the most general quantum mechanical observables. The following summary collects their properties insofar as far as they are relevant here, and some notation will be established.

Consider a quantum system capable of residing in d states $|\psi_n\rangle$, $n = 1 \ldots d$, which form an orthonormal basis of the d-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$. A hermitean operator $\hat{E} = \hat{E}^\dagger$ is called *positive semi-definite*, $\hat{E} \geq 0$, if there is no state which produces a negative expectation value for $\hat{E}$, or equivalently,

$$\langle \psi_n | \hat{E} | \psi_n \rangle \geq 0 \,, \quad n = 1 \ldots d \,. \tag{1}$$

The density matrix $\hat{\rho}$ used to describe a mixed state of a quantum system provides a well-known example of such an operator, $\hat{\rho} \geq 0$. A collection of positive operators $\hat{E}_\alpha, \alpha \in A$, with A being a discrete or continuous set of labels,

qualifies as a *positive operator-valued measure* in $\mathcal{H}$, or POVM for short, if its elements sum up to the identity in $\mathcal{H}$,

$$\sum_{\alpha \in A} \hat{E}_\alpha = \hat{\mathbb{1}} \, . \tag{2}$$

When α is a continuous label, the symbol $\sum$ is understood to denote an integration over A. Taking the expectation value of this equation in any normalized state $|\psi\rangle$, one finds that the discrete or continuous set of positive numbers $p_\alpha = \langle \psi | \hat{E}_\alpha | \psi \rangle, \alpha \in A$, sum up to one. Thus, the numbers p_α have the properties of a probability distribution which suggests to think of the operators $\hat{E}_\alpha, \alpha \in A$, as an "operator-valued" *measure.*

Here are four examples of POVMs. The first example consists of only one element, the identity $\hat{\mathbb{1}}$ in $\mathbb{C}^d$. Next, the completeness relation of the states $|\psi_n\rangle$,

$$\sum_{n=1}^{d} |\psi_n\rangle\langle\psi_n| = \hat{\mathbb{1}} \tag{3}$$

shows that the collection of the positive semi-definite, orthonormal projectors $\hat{E}_n \equiv |\psi_n\rangle\langle\psi_n|$ form a POVM with d elements.

A POVM may contain any number of elements, not restricted by d, the dimension of the underlying Hilbert space. In such a situation, the elements of the POVM cannot consist of orthonormal projections since the space $\mathcal{H}$ accommodates at most d orthogonal states. Consider the example of a POVM for a qubit with Hilbert space $\mathbb{C}^2$, defined in terms of the states $|\pm\rangle$, the eigenstates of the z-component of a spin $1/2$, equivalent to the computational basis for the qubit. It consists of *three* operators,

$$\hat{E}_1 = \frac{\sqrt{2}}{1 + \sqrt{2}} |-\rangle\langle-| \, , \quad \hat{E}_2 = \frac{\sqrt{2}}{1 + \sqrt{2}} \left(|-\rangle - |+\rangle \right) \left(\langle-| - \langle+| \right) \, , \tag{4}$$

and $\hat{E}_3 = \hat{\mathbb{1}} - \hat{E}_1 - \hat{E}_2$, which sum up to the identity. This POVM allows one to successfully differentiate between non-orthogonal quantum states [9]. Imagine that you are being asked to find out whether you have been sent the state $|?\rangle$ which could be either $|+\rangle$ or $(1/\sqrt{2})(|-\rangle + |+\rangle)$. Using the above POVM to perform a measurement on the unknown state $|?\rangle$, you will find, in each run, an outcome associated with one of the three operators given above. In the first case, associated with $\hat{E}_1$, you know that the state provided cannot have been $|+\rangle$ since $\langle+|\hat{E}_1|+\rangle = 0$; similarly, you know that the unknown state must have been $|+\rangle$ if the measurement outcome corresponds to $\hat{E}_2$ since only this state has a non-zero component "along" $\hat{E}_2$. If the third outcome occurs, nothing can be said about $|?\rangle$. If you were to perform a measurement with any two *orthonormal* projections, you could draw no conclusions about $|?\rangle$ from a single run. By invoking the POVM defined in (4), however, it is possible to extract the desired information from a *single* run of an experiment if either outcome 1 or 2 occur.

The final example of a POVM has uncountably many elements: let

$$\hat{E}_{\mathbf{n}} = |\mathbf{n}\rangle\langle\mathbf{n}| \, , \quad |\mathbf{n}\rangle \in \mathcal{S} \, , \tag{5}$$

where $|\mathbf{n}\rangle$ is a coherent state of a spin $s \equiv (d-1)/2$, the label $\mathbf{n}$ being a vector pointing from the origin to the point $P_{\mathbf{n}}$ on the unit sphere $\mathcal{S}$ in $\mathbb{R}^3$. The overcompleteness relation of the coherent states [10] implies that these operators are a indeed a POVM,

$$\int_{\mathcal{S}} \hat{E}_{\mathbf{n}} \, d\mu(\mathbf{n}) \equiv \frac{d}{4\pi} \int_{\mathcal{S}} |\mathbf{n}\rangle\langle\mathbf{n}| \, d\mathbf{n} = \hat{\mathbb{1}} \, . \tag{6}$$

B. Minimal informationally complete POVMs

If a POVM is to be *informationally complete* (IC for short), each set of probabilities $\rho_\alpha, \alpha \in A$, must identify a unique density matrix $\hat{\rho}$ satisfying

$$\rho_\alpha \equiv \text{Tr}\left[\hat{\rho} \hat{E}_\alpha \right] \, . \tag{7}$$

The first three examples of POVMs just described in the previous section are not informationally complete while the coherent-state POVM defined in (6) is: only one operator $\hat{A}$ is associated with a Q-symbol $A_{\mathbf{n}} = \langle \mathbf{n} | \hat{A} | \mathbf{n} \rangle \equiv \text{Tr}[\hat{A} \hat{E}_{\mathbf{n}}]$ [10].

Minimal informationally complete POVMs, or MIC-POVMs, contain the *least* number of elements such that the probabilities in (7) determine a unique density matrix $\hat{\rho}$. This requirement is equivalent to saying that the operators $\hat{E}_\alpha$ form a (minimal) *basis* of the vector space of hermitean operators acting on the Hilbert space $\mathbb{C}^d$. Counting the number of real parameters necessary to parameterize all such operators, conveniently represented as hermitean matrices of size $(d \times d)$, one concludes that a MIC-POVMs will contain precisely d^2 (linearly independent) elements. For convenience, the normalization condition $\mathrm{Tr}[\hat{\rho}] = 1$ is often relaxed, so that density matrices are indeed parameterized by a total of d^2 real numbers.

Not every set of d^2 operators spanning the hermitean operators on $\mathbb{C}^d$ is a POVM. To see this, let us look at the example of a spin $1/2$, or qubit. Observables $\hat{A}$ have the form

$$\hat{A} = A_0 \hat{\mathbb{I}} + \mathbf{A} \cdot \hat{\boldsymbol{\sigma}} \,, \tag{8}$$

with a real number A_0 and a real three-component vector $\mathbf{A}$, while $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ denotes the spin operator. The four operators $(\hat{\mathbb{I}}, \hat{\boldsymbol{\sigma}})$ do not constitute a POVM since the expectation values of each of the operators $\hat{\sigma}_i, i = x, y, z$, range from -1 to $+1$. However, all is not lost yet: the three indefinite operators turn positive by adding the identity:

$$0 \le \langle \psi | \left(\hat{\mathbb{I}} + \hat{\sigma}_i \right) | \psi \rangle \le 2 \,, \quad i = x, y, z \,. \tag{9}$$

This observation makes it easy to construct examples of MIC-POVMs for a qubit,

$$\hat{E}_n = \frac{1}{4} \left(\hat{\mathbb{I}} + \mathbf{n}_n \cdot \hat{\boldsymbol{\sigma}} \right) \ge 0 \,, \quad n = 1 \ldots 4, \quad \text{where} \quad \sum_{n=1}^{4} \mathbf{n}_n = 0 \,, \tag{10}$$

and the four unit vectors $\mathbf{n}_n$ must not lie in a plane [11],.

The CFS-construction presented in [12] ascertains the existence of MIC-POVMs for qudits living in $\mathbb{C}^d$. Consider any set of d^2 linearly independent positive definite operators $\hat{F}_\alpha > 0$, say, satisfying the relation

$$\sum_{\alpha=1}^{d^2} \hat{F}_\alpha = \hat{G} > 0 \,. \tag{11}$$

Being positive definite, the operator $\hat{G}$ has a unique, strictly positive square root $\hat{G}^{\frac{1}{2}}$ the inverse of which, $\hat{G}^{-\frac{1}{2}}$, exists as well. Thus, the transformation $\hat{F}_\alpha \to \hat{E}_\alpha = \hat{G}^{-\frac{1}{2}} \hat{F}_\alpha \hat{G}^{-\frac{1}{2}}$ is invertible, preserves positivity, hermiticity and the rank of the original operators. What is more, the new operators satisfy the relation (2) with $A = \{1 \ldots d^2\}$, thus giving rise to a MIC-POVM. As shown in [12] there is at least one collection of d^2 linearly independent positive definite operators in $\mathbb{C}^d$.

Particularly interesting examples of MIC-POVMs consist of projection operators onto d^2 states $|\varphi_n\rangle, n = 1 \ldots d^2$, such that their pairwise scalar products are of modulus $1/(d+1)$,

$$|\langle \varphi_n | \varphi_{n'} \rangle|^2 = \frac{1}{d+1} \,, \quad n \ne n' \,. \tag{12}$$

It is known how to analytically construct such *symmetric* MIC-POVMs, or SIC-POVMs for short, in some Hilbert spaces of small dimensions as well as $d = 19$ [13], although numerical evidence up to $d = 45$ [14] seems to suggest that they exist in any dimension (see [15] for a survey).

III. THE EXPECTATION-VALUE REPRESENTATION OF QUANTUM MECHANICS

A. Definition of the Expectation-Value Representation

If you randomly pick d^2 points $\mathbf{n}_n, n = 1 \ldots d^2$, on the unit sphere, then the operators $\hat{Q}_n = |\mathbf{n}_n\rangle\langle\mathbf{n}_n|$, projecting on the associated coherent states $|\mathbf{n}_n\rangle$ are, with probability one, linearly independent [16]. Consequently, they provide a basis for the hermitean operators on $\mathbb{C}^d$,

$$\hat{A} = \frac{1}{d} \sum_{n=1}^{d^2} A^n \hat{Q}_n \,, \tag{13}$$

with unique real coefficients A^n (different from $\text{Tr}[\hat{A}\hat{Q}_n]$). The trace of the product of two operators on $\mathbb{C}^d$ defines a scalar product (one needs to invoke the adjoint of one of the operators if non-hermitean operators are considered) which can be used to introduce a second basis *dual* to the projectors $\hat{Q}_n$,

$$\frac{1}{d}\text{Tr}\left[\hat{Q}^n\hat{Q}_{n'}\right] = \delta_{n'}^n, \quad n, n' = 1\dots d^2. \tag{14}$$

The dual operators $\hat{Q}^n, n = 1\dots d^2$, provide a basis for observables just as the original ones do,

$$\hat{A} = \frac{1}{d}\sum_{n=1}^{d^2} A_n\hat{Q}^n, \tag{15}$$

with a second set of real expansion coefficients A_n. Using (14), one sees that the expansion coefficients in one basis are given by the scalar product of the operator at hand with the corresponding element of the dual basis,

$$A^n = \text{Tr}\left[\hat{A}\hat{Q}^n\right], \quad \text{and} \quad A_n = \text{Tr}\left[\hat{A}\hat{Q}_n\right], \quad n = 1\dots d^2. \tag{16}$$

It is interesting to point out that the coefficients A_n and A^n can be thought of as discrete, non-redundant versions of the Q- and P-symbols of the operator $\hat{A}$, respectively [16]. Knowing one set of coefficients, the other set is determined uniquely by

$$A_n = \frac{1}{d}\sum_{m=1}^{d^2} \mathfrak{G}_{nm}A^m, \tag{17}$$

where $\mathfrak{G}$ is the non-singular Gram matrix of the basis $\hat{Q}_n$, with elements $\mathfrak{G}_{nm} = \text{Tr}[\hat{Q}_n\hat{Q}_m]$. The coefficients A_n have a simple physical meaning: recalling that the $\hat{Q}_n$ are projections, one has

$$A_n = \langle\mathbf{n}_n|\hat{A}|\mathbf{n}_n\rangle, \tag{18}$$

saying that any operator $\hat{A}$ is determined entirely by its expectation values in d^2 appropriate coherent states. Consequently, it is possible to parameterize the density matrix $\hat{\rho}$ of a qudit (or a spin with $s = (d-1)/2$) by d^2 probabilities, $p_n = \langle\mathbf{n}_n|\hat{\rho}|\mathbf{n}_n\rangle$. These probabilities can be measured in d^2 independent experiments, each corresponding to a different orientation of a standard Stern-Gerlach apparatus [7]. When expressing a density matrix by means of (15) $\hat{\rho}$ is said to be given in the *expectation-value representation* (EVR for short).

B. Obstacles

Let us now explore whether the expectation-value representation gives rise to MIC-POVMs. Being positive semi-definite, the d^2 operators $\hat{Q}_n$ are promising candidates for a minimal informationally POVM. But do they add up to the identity? There is an expansion of the identity,

$$\hat{\mathbb{I}} = \frac{1}{d}\sum_{n=1}^{d^2} \mathbb{I}^n\hat{Q}_n, \tag{19}$$

and the rescaled projectors $(\mathbb{I}^n/d)\hat{Q}_n$ would constitute a MIC-POVM if all coefficients $\mathbb{I}^n = \text{Tr}[\hat{Q}^n]$ were known to be positive. Unfortunately, the numbers $\mathbb{I}^n$ are *not* guaranteed to be positive for all constellations of directions $\{\mathbf{n}_n, n = 1\dots d^2\}$, as follows from the example presented in Section V B.

In view of this result, it might be a good idea to expand the identity in the *dual* basis,

$$\hat{\mathbb{I}} = \frac{1}{d}\sum_{n=1}^{d^2} \hat{Q}^n, \tag{20}$$

with automatically positive expansion coefficients $\mathbb{I}_n = \langle\mathbf{n}_n|\hat{\mathbb{I}}|\mathbf{n}_n\rangle \equiv 1$. However, some of the dual operators $\hat{Q}^n$ will, in general, not be positive semi-definite. To see this, consider the elements of Gram matrix $\mathfrak{G}$ which are non-negative,

$$\mathfrak{G}_{nn'} = \text{Tr}\left[\hat{Q}_n\hat{Q}_{n'}\right] = |\langle\mathbf{n}_n|\mathbf{n}_{n'}\rangle|^2 \geq 0, \quad n, n' = 1\dots d^2; \tag{21}$$

the value zero is attained only if two vectors happen to point to diametrically opposite points, $\mathbf{n}_n = -\mathbf{n}_{n'}$, implying that the Gram matrix has at most one zero in each row. It follows that the inverse $\mathfrak{G}^{-1}$ of the Gram matrix must have at least one *negative* entry (actually, in each row): the off-diagonal elements of the product $\mathfrak{G}^{-1}\mathfrak{G}$ could not vanish otherwise. Expressing the matrix elements of $\mathfrak{G}^{-1}$ by the scalar products of the elements of the dual basis, one is led to conclude that

$$\mathfrak{G}^{\nu\nu'} = \mathrm{Tr}\left[\hat{Q}^\nu \hat{Q}^{\nu'}\right] < 0\,, \tag{22}$$

for at least one pair of indices ν, ν', say. This relation is incompatible with all operators $\hat{Q}^n, n = 1 \ldots d^2$, being positive semi-definite: evaluate the trace in (22) in the eigenstates $|Q_r^\nu\rangle, r = 1 \ldots d$, of the operator $\hat{Q}^\nu$ with eigenvalues Q_r^ν. This implies

$$\mathrm{Tr}\left[\hat{Q}^\nu \hat{Q}^{\nu'}\right] = \sum_{r=1}^{d} Q_r^\nu \langle Q_r^\nu | \hat{Q}^{\nu'} | Q_r^\nu \rangle < 0\,, \tag{23}$$

which requires at least one negative term in the sum. Consequently, either $\hat{Q}^\nu$ must have a negative eigenvalue or there is a state such that the expectation value of $\hat{Q}^{\nu'}$ is negative. Both alternatives show that not all dual operators $\hat{Q}^\nu$ can be positive semi-definite. Thus, one of the two operators in (22) is not be positive semi-definite, and the relation (20) does *not* define a POVM.

IV. CONSTRUCTING NEW MIC-POVMS

The minimal informationally complete sets $\{\hat{Q}_n, n = 1 \ldots d^2\}$ and $\{\hat{Q}^n, n = 1 \ldots d^2\}$ will serve as starting points to construct new MIC-POVMs.

A. CFS-construction

Let us apply the method by CFS to construct a POVM out of positive multiples of the projection operators $\hat{Q}_n$. The sum

$$\hat{S} = \sum_{n=1}^{d^2} \alpha_n \hat{Q}_n \,, \alpha_n > 0\,, \tag{24}$$

defines a hermitean, *strictly* positive operator, $\hat{S} > 0$, as shown now. The expectation value of $\hat{S}$ in a state $|\psi\rangle$ is clearly non-negative, $\langle \psi | \hat{S} | \psi \rangle \geq 0$; equivalently, its eigenvalues are non-negative throughout. However, $\hat{S}$ having a *zero* eigenvalue would lead to a contradiction: assume that there is a normalizable state $|\psi_0\rangle$ which $\hat{S}$ annihilates, $\hat{S}|\psi_0\rangle = 0$, and expand the associated projector $\hat{S}_0 = |\psi_0\rangle\langle\psi_0|$ in terms of the basis $\hat{Q}^n$. The sum of the non-negative expansion coefficients

$$S_n = \mathrm{Tr}[\hat{S}_0 \hat{Q}_n] = |\langle\psi_0|\mathbf{n}_n\rangle|^2 \tag{25}$$

would vanish since one has

$$\sum_{n=1}^{d^2} \alpha_n S_n = \langle\psi_0| \sum_{n=1}^{d^2} \alpha_n \hat{Q}_n |\psi_0\rangle = \langle\psi_0| \left(\hat{S}|\psi_0\rangle\right) = 0\,, \tag{26}$$

which is only possible if each term S_n of the sum vanishes individually. Hence, $\hat{S}_0$ must be zero, contradicting the assumption that $|\psi_0\rangle$ is normalizable state. This leaves us with $\hat{S} > 0$, and the operator $\hat{S}$ thus has a unique square root and an inverse, which is sufficient to complete the CFS-construction. Explicitly, the resulting family of MIC-POVMs is given by

$$\{\hat{E}_n = \alpha_n \hat{S}^{-\frac{1}{2}} \hat{Q}_n \hat{S}^{-\frac{1}{2}} \,, \alpha_n > 0 \,, n = 1 \ldots d^2\}\,. \tag{27}$$

As no analytic expressions for the square roots are available beyond $d = 4$, the POVM just constructed will in general not be in closed form.

B. MIC-POVMs from the EVR: first case

As they stand, the expansions given in Eqs. (19) and (20) do not define POVMs since neither the expansion coefficients $\mathbb{I}^n$ nor the elements of the basis $\hat{Q}^n$ are generally non-negative. It will be shown now that minor modifications are sufficient in order to obtain MIC-POVMs.

Rearrange the terms in (19) in such a way that a first sum contains expressions with non-negative coefficients only, $\mathbb{I}^{n+} \geq 0$, while a second sum combines the terms with $\mathbb{I}^{n-} < 0$,

$$\hat{\mathbb{I}} = \frac{1}{d} \sum_{n_+=1}^{N_+} \mathbb{I}^{n+} \hat{Q}_{n_+} - \frac{1}{d} \sum_{n_-=1}^{N_-} |\mathbb{I}^{n-}| \hat{Q}_{n_-}, \quad N^+ + N^- = d^2 . \tag{28}$$

Add a (C/d)-fold multiple of the identity on both sides, with $C = \sum_{n_-} |\mathbb{I}^{n-}| > 0$, to find

$$\left(1 + \frac{C}{d}\right) \hat{\mathbb{I}} = \frac{1}{d} \sum_{n_+=1}^{N_+} \mathbb{I}^{n+} \hat{Q}_{n_+} + \frac{1}{d} \sum_{n_-=1}^{N_-} |\mathbb{I}^{n-}| (\hat{\mathbb{I}} - \hat{Q}_{n_-}) . \tag{29}$$

This can be written as

$$\hat{\mathbb{I}} = \sum_{n_+=1}^{N_+} \hat{E}_{n_+} + \sum_{n_-=1}^{N_-} \hat{E}_{n_-} \equiv \sum_{n=1}^{d^2} \hat{E}_n , \tag{30}$$

with d^2 positive semi-definite operators, N_+ of which have rank one and N_- have rank $(d-1)$,

$$\hat{E}_{n_+} = \frac{\mathbb{I}^{n+}}{d+C} \hat{Q}_{n_+} \geq 0 , \quad \hat{E}_{n_-} = \frac{|\mathbb{I}^{n-}|}{d+C} (\hat{\mathbb{I}} - \hat{Q}_{n_+}) \geq 0 . \tag{31}$$

Due to (30), the operators $\hat{E}_{n_\pm}$ form a MIC-POVM, having a simple physical interpretation: this POVM has d^2 possible outcomes, N_+ of which correspond to finding the system in one of the coherent states $|\mathbf{n}_{n_+}\rangle$, while the remaining N_- outcomes indicate that it is in a state with non-zero component in a $(d-1)$-dimensional subspace orthogonal to one of the states $|\mathbf{n}_{n_-}\rangle$. The case of a qubit is special since the operators $\hat{E}_{n_\pm}$ are of rank one throughout. The MIC-POVM in (30) is given in closed form for any dimension d.

C. MIC-POVMs from the EVR: second case

Not surprisingly, similar modifications enable one to construct a MIC-POVM out of the elements $\hat{Q}^n$ of the dual basis. Write the expansion (20) of the identity as

$$\left(1 + \frac{\tilde{C}}{d}\right) \hat{\mathbb{I}} = \frac{1}{d} \sum_{n_+=1}^{\tilde{N}_+} \hat{Q}^{n+} + \frac{1}{d} \sum_{n_-=1}^{\tilde{N}_-} \left(\hat{Q}^{n-} + |q^{n-}| \hat{\mathbb{I}}\right) , \tag{32}$$

where the first sum contains positive semi-definite operators only, and the second one takes care of the indefinite ones; the number $q^{n-} < 0$ denotes the smallest eigenvalue of $\hat{Q}^{n-}$ and $\tilde{C}$ is the sum of their moduli,

$$\tilde{C} = \sum_{n_-=1}^{\tilde{N}_-} |q^{n-}| > 0 . \tag{33}$$

Then, the operators

$$\hat{\varepsilon}^{n+} = \frac{1}{d+\tilde{C}} \hat{Q}^{n+} , \quad \hat{\varepsilon}^{n-} = \frac{1}{d+\tilde{C}} (\hat{Q}^{n-} + |q^{n-}| \hat{\mathbb{I}}) , \tag{34}$$

are positive semi-definite by construction and give rise to a POVM,

$$\hat{\mathbb{I}} = \sum_{n_+=1}^{\tilde{N}_+} \hat{\varepsilon}^{n+} + \sum_{n_-=1}^{\tilde{N}_-} \hat{\varepsilon}^{n-} \equiv \sum_{n=1}^{d^2} \hat{\varepsilon}^n . \tag{35}$$

As Eq. (34) involves the smallest eigenvalues of some operators, the resulting POVM is not given in closed form. It is not difficult to see that this MIC-POVMs is not dual to the one constructed in the previous section: taking the scalar products within each basis one has

$$\text{Tr}\left[\hat{E}_n\hat{E}_{n'}\right] \geq 0\,, \quad \text{and} \quad \text{Tr}[\hat{\varepsilon}^n\hat{\varepsilon}^{n'}] \geq 0\,, \quad n,n' = 1\ldots N\,. \tag{36}$$

Thus, both sets of operators define their own Gram matrices with only non-negative entries only; not being diagonal, these matrices cannot be inverse to each other. The MIC-POVMs in (30) and (35) are intrinsically different.

D. General MIC-POVMs

Having gained some experience with the construction of MIC-POVMs, it becomes obvious how to generalize the CFS-approach. Effectively, one can both relax the condition of having d^2 *non-negative* operators and avoid the appearance of the analytically inaccessible square root of an operator. Explicitly, it will be shown that *every set of d^2 linearly independent hermitean operators acting on $\mathbb{C}^d$ can be used to define a closed-form MIC-POVM*.

Consider d^2 hermitean operators $\hat{\kappa}_n$ on $\mathbb{C}^d$ with extremal eigenvalues $-\infty < \kappa_n^{\pm} < \infty$, not both of which can be equal to zero simultaneously. Since they satisfy the inequalities

$$\kappa_n^- \leq \hat{\kappa}_n \leq \kappa_n^+\,, \quad n = 1\ldots d^2\,, \tag{37}$$

the shifted and rescaled operators

$$\hat{K}_n = \frac{1}{\kappa_n^+ - \kappa_n^-}\left(\hat{\kappa}_n - \kappa_n^-\hat{\mathbb{1}}\right)\,, \quad n = 1\ldots d^2\,, \tag{38}$$

are bounded by zero and one,

$$0 \leq \hat{K}_n \leq 1\,, \quad n = 1\ldots d^2\,, \tag{39}$$

as is necessary for the elements of a POVM. The conditions

$$\frac{1}{d}\text{Tr}\left[\hat{K}^{n'}\hat{K}_n\right] = \delta_n^{n'} \tag{40}$$

determine a unique dual set of d^2 operators, $\hat{K}^n$. Hence, there are two expansions of the identity,

$$\hat{\mathbb{1}} = \frac{1}{d}\sum_{n=1}^{d^2}\mathbb{1}^n\hat{K}_n = \frac{1}{d}\sum_{n=1}^{d^2}\mathbb{1}_n\hat{K}^n\,, \tag{41}$$

where

$$\mathbb{1}^n = \text{Tr}\left[\hat{\mathbb{1}}\hat{K}^n\right]\,, \quad \text{and} \quad \mathbb{1}_n = \text{Tr}\left[\hat{\mathbb{1}}\hat{K}_n\right]\,. \tag{42}$$

As before, some of the coefficients $\mathbb{1}^n$ may be negative and the dual operators $\hat{K}^n$ are not necessarily positive semi-definite. In the first case, follow the procedure described in Sec. IV B: effectively replace the operators $\hat{K}_{n_-}$ (the ones with *negative coefficients*) by $(\hat{\mathbb{1}} - \hat{K}_{n_-})$ leading to set of d^2 non-negative operators which sum up to the identity

$$\hat{\mathbb{1}} = \sum_{n_+=1}^{N_+}\hat{\epsilon}_{n_+} + \sum_{n_-=1}^{N_-}\hat{\epsilon}_{n_-} \equiv \sum_{n=1}^{d^2}\hat{\epsilon}_n\,, \tag{43}$$

where

$$\hat{\epsilon}_{n_+} = \frac{\mathbb{1}^{n_+}}{d+C}\hat{K}_{n_+}\,, \quad \hat{\epsilon}_{n_-} = \frac{|\mathbb{1}^{n_-}|}{d+C}\left(\hat{\mathbb{1}} - \hat{K}_{n_-}\right)\,, \tag{44}$$

C being defined as the sum of the moduli of the negative coefficients.

In the second case, the indefinite dual operators can be made positive semi-definite by adding an appropriate multiple of the identity to (41), in complete analogy to the procedure presented in Sec. IV C. As a result, it has been shown that at least two different MIC-POVMs can be introduced given any set of d^2 linearly independent hermitean operators.

For $d > 4$, no analytic expressions for the extremal eigenvalues $\kappa_n^\pm$ of the operators $\hat{\kappa}_n$ exist in general. If a MIC-POVM in closed form is required, one can resort to the weaker inequalities

$$-\|\hat{\kappa}_n\| \leq \hat{\kappa}_n \leq \|\hat{\kappa}_n\|, \quad n = 1 \ldots d^2, \tag{45}$$

with any matrix norm $\|\hat{M}\|$. The advantage is that $\|\hat{\kappa}_n\|$ can be calculated explicitly once the matrix elements of $\hat{\kappa}_n$ are known in some basis. Subsequently, one obtains a modified version of Eq. (38),

$$\hat{K}_n = \frac{1}{2\|\hat{\kappa}_n\|} \left(\hat{\kappa}_n + \|\hat{\kappa}_n\|\hat{\mathbb{I}} \right), \quad n = 1 \ldots d^2, \tag{46}$$

ending up with a MIC-POVM given *in closed form*, for any set of d^2 linearly independent operators $\hat{\kappa}_n$.

V.   EXAMPLES: MIC-POVMS FOR A QUBIT

A.   Tetrahedral MIC-POVM

Consider the operators $\hat{Q}_n = |\mathbf{n}_n\rangle\langle\mathbf{n}_n|, n = 1 \ldots 4$, defined by the following four vectors,

$$\mathbf{n}_1 = (0, 0, 1), \ \mathbf{n}_2 = \frac{1}{3}\left(2\sqrt{2}, 0, -1\right), \ \mathbf{n}_3 = \frac{1}{3}\left(-\sqrt{2}, \sqrt{6}, -1\right), \ \mathbf{n}_4 = \frac{1}{3}\left(-\sqrt{2}, -\sqrt{6}, -1\right), \tag{47}$$

which point to the vertices of a tetrahedron. The projectors $\hat{Q}_n$ are linear independent since their Gram matrix is invertible,

$$\mathfrak{G} = \frac{1}{3}\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \quad \mathfrak{G}^{-1} = \frac{1}{4}\begin{pmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix}. \tag{48}$$

The dual operators are given by $\hat{Q}^n = (1/2)\sum_m \text{Tr}[\hat{Q}^n\hat{Q}^m]\hat{Q}_m$, which gives

$$\hat{Q}^1 = \frac{1}{2}\left(5\hat{Q}_1 - \hat{Q}_2 - \hat{Q}_3 - \hat{Q}_4\right), \quad \text{etc.}, \tag{49}$$

recalling that $\text{Tr}[\hat{Q}^n\hat{Q}^m] \equiv d^2\mathfrak{G}^{nm}$, where $\mathfrak{G}^{nm} = [\mathfrak{G}^{-1}]_{nm}$. This leads to

$$\hat{\mathbb{I}} = \frac{1}{2}\sum_{n=1}^{4} \hat{Q}_n, \tag{50}$$

where $\mathbb{I}^n = \text{Tr}[\hat{Q}^n] = (1/2)(5 - 1 - 1 - 1) = 1, n = 1 \ldots 4$, has been used. Note that the discrete P- and Q-symbol of the identity coincide and are both positive, $\mathbb{I}_n = \mathbb{I}^n = 1$. The resulting operators $\hat{E}_n = \hat{Q}_n/2 \equiv (1/4)(\hat{\mathbb{I}} + \mathbf{n}_n \cdot \hat{\boldsymbol{\sigma}})$ are exactly those given in Eq. (10), thus constituting a MIC-POVM and even a SIC-POVM [11].

The dual operators $\hat{Q}^n$ are not positive semi-definite: consider the expectation value of $\hat{Q}^1$ in Eq. (49) in the state $|-\mathbf{n}_1\rangle$, for example,

$$\langle-\mathbf{n}_1|\hat{Q}^1|-\mathbf{n}_1\rangle = \frac{1}{4}(-3 + \mathbf{n}_1 \cdot \mathbf{n}_2 + \mathbf{n}_1 \cdot \mathbf{n}_3 + \mathbf{n}_1 \cdot \mathbf{n}_4) < 0, \tag{51}$$

where $|\langle\mathbf{n}|\mathbf{n}'\rangle|^2 = (1/2)(1 + \mathbf{n} \cdot \mathbf{n}')$ has been used. Consequently, one would need to apply the procedure described in Sec. IV C to determine a second MIC-POVM.

B.   Generic MIC-POVMs

This section gives an example of the expectation-value representation for a qubit where the expansion coefficients of the identity with respect to the original basis are not all positive. Let us consider three pairwise orthogonal unit vectors $\mathbf{n}_i$, $i = 1, 2, 3$, in $\mathbb{R}^3$, and

$$\mathbf{n}_4 = \frac{1}{\sqrt{3}} \left(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 \right) . \tag{52}$$

These four vectors clearly do not sum up to zero as is required in (10), hence the four projection operators $\hat{Q}_n = |\mathbf{n}_n\rangle\langle\mathbf{n}_n|$, $n = 1 \ldots 4$, do not form a POVM. They are, however, linear independent since the vectors $\mathbf{n}_n$ do not lie on a cone which, according to [17] is sufficient for a non-singular Gram matrix.

The procedure outlined in Sec. IV C associates with them a unique MIC-POVM,

$$\hat{E}_i = \frac{2}{\sqrt{3}(\sqrt{3}+1)} |\mathbf{n}_i\rangle\langle\mathbf{n}_i| , \quad i = 1, 2, 3 , \tag{53}$$

$$\hat{E}_4 = \frac{2}{(\sqrt{3}+1)} |-\mathbf{n}_4\rangle\langle -\mathbf{n}_4| . \tag{54}$$

The derivation of this result is simplified by the the fact [17] that one can express the expansion coefficients of the identity in the form

$$\mathbb{I}^n = \frac{4}{1 + \mathbf{f}^n \cdot \mathbf{n}_n} , \quad n = 1 \ldots 4 , \tag{55}$$

where the vector $\mathbf{f}^1 \in \mathbb{R}^3$ is determined by

$$\mathbf{f}^1 = -\frac{\mathbf{n}_2 \wedge \mathbf{n}_3 + \mathbf{n}_3 \wedge \mathbf{n}_4 + \mathbf{n}_4 \wedge \mathbf{n}_2}{(\mathbf{n}_2 \wedge \mathbf{n}_3) \cdot \mathbf{n}_4} , \tag{56}$$

and the other three vectors follow from this relation by cyclic permutation of the indices 1 through 4. A straightforward but still lengthy calculation leads to

$$\mathbb{I}^i = \frac{4}{\sqrt{3}(\sqrt{3}-1)} > 0 , \quad i = 1, 2, 3 , \tag{57}$$

$$\mathbb{I}^4 = \frac{4}{(1-\sqrt{3})} < 0 , \tag{58}$$

i.e. there is one negative coefficient which needs to be eliminated by adding a multiple of the identity to the expansion of the identity. Apply now the method outlined in Sec. IV B, and you will find the MIC-POVM specified in (53,54). A second MIC-POVM could be obtained from the dual basis but no further insight is to be gained form its explicit form.

VI.   SUMMARY AND CONCLUSIONS

Starting from the expectation-value representation of quantum mechanics in d-dimensional Hilbert spaces, new simple POVMs with d^2 elements have been introduced which are informationally complete. Mathematically speaking, the elements of these POVMs provide a basis in the Hilbert-Schmidt space of operators acting on $\mathbb{C}^d$ while, from a physical point of view, they are suited to reconstruct unknown quantum states if an arbitrarily large number of systems in the same state are available. Repeated measurements with such a POVM generate d^2 probabilities which are in a one-to-one correspondence with a density matrix $\hat{\rho}$. Since any set of d^2 linearly independent operators can be used as a starting point, a wide range of possibilities opens up to construct MIC-POVMs most suited for the application at hand.

It seems worthwhile to finally point out how to explicitly write down a density matrix $\hat{\rho}$ once the probabilities

$$p_n(\hat{\rho}) = \mathrm{Tr}\left[\hat{\rho}\hat{E}_n\right] \in [0, 1] , \quad n = 1 \ldots d^2 , \tag{59}$$

associated with a MIC-POVM $\hat{E}_n, n = 1 \ldots d^2$, have been measured. The most direct approach invokes the dual operators $\hat{E}^n$, defined by the equivalent of the condition (14). Once these operators have been found, the density matrix is given by

$$\hat{\rho} = \frac{1}{d} \sum_{n=1}^{d^2} p_n(\hat{\rho}) \hat{E}^n \,. \tag{60}$$

Formally, this result is very similar to Eq. (15), its equivalent in the expectation-value representation. However, there is a fundamental difference since the numbers $p_n(\hat{\rho})$ are 'honest' probabilities emerging from an experiment performed with a *single* apparatus while the probabilities required for the expectation-value representation are obtained from running d^2 different experiments.

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